Coverings and dimensions in infinite profinite groups

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Abstract

Answering a question of Miklós Abért, we prove that an infinite profinite group cannot be the union of less than continuum many translates of a compact subset of box dimension less than 1. Furthermore, we show that it is consistent with the axioms of set theory that in any infinite profinite group there exists a compact subset of Hausdorff dimension 0 such that one can cover the group by less than continuum many translates of it.[1]

Introduction

Recently one can find papers concerning the following questions. If a compact subset of reals is small in a sense, does it follow that the real line cannot be the union of less than continuum many translates of it?

Of course, the answer depends on our notion "small". U. B. Darji and T. Keleti [4] proved that if a compact subset $C \subseteq \mathbb{R}$ has packing dimension (or box dimension) less than 1, then $\mathbb{R}$ is not the union of less than continuum many translates of C.

Assuming the continuum hypothesis, if C has Lebesgue measure zero, then $\mathbb{R}$ can only be covered with continuum many translates of C. However, M. Elekes and J. Steprans [5] showed that in a model of $\mathrm{ZFC} + cof(\mathcal{N}) < 2^\omega$, there exists a compact set $C \subseteq \mathbb{R}$ of Lebesgue measure zero such that $\mathbb{R}$ can be covered with less than continuum many translates of C, where $cof(\mathcal{N})$ stands for the cofinality of Lebesgue nullsets. The existence of such a model is consistent with ZFC, fix such a model $\mathcal{M}$ for our purpose.

M. Elekes and Á. Tóth [6] proved the analogous statement for commutative Polish groups (with Haar measure taking the role of Lebesgue measure). They also

[1] The material covered here was part of the author's Master's Thesis at Eötvös University, Budapest; supervisor: Tamás Keleti

reduced the noncommutative case to the investigation of Lie groups and profinite groups.

M. Abért [1] proved the theorem for infinite profinite groups. In the same article he asked whether similar theorems to the result of Darji and Keleti can hold in the case of infinite profinite groups and box dimension: is it true that if a compact subset has box dimension less than 1, then the group cannot be covered with less than continuum many translates of it?

In this paper we prove the following results. First, applying the method due to Darji and Keleti, we show that if a compact subset of an infinite profinite group has box dimension less than 1, then the group is not the union of less than continuum many translates of it.

Then, we show that in $\mathcal{M}$, any infinite profinite group contains a set of Hausdorff-dimension 0 such that the group is the union of less than continuum many translates of it. Our technique is based on A. Máthé's [8], who proved the theorem in the case of reals.

Note that although the two notions of dimension differ, they coincide in the case of closed subgroups [2].

We will use the following definition of profinite groups (which ensures that the group is infinite).

Definition. *Let $G_1, G_2, \ldots$ be finite groups and let $\varphi_k : G_{k+1} \to G_k$ be onto, but not one-to-one homomorphisms. Then they determine the profinite group G, whose elements are*

$$\{(g_1, g_2, \ldots) \mid \forall k \in \mathbb{N} : g_k \in G_k, \varphi_k(g_{k+1}) = g_k\},$$

while the multiplication of G is defined as the product of the multiplication in coordinates:

$$(g_1, g_2, \ldots) \cdot_G (h_1, h_2, \ldots) = (g_1 \cdot_{G_1} h_1, g_2 \cdot_{G_2} h_2, \ldots).$$

For any $x \in G$, we denote by $x_{|k}$ the k-th coordinate of x. For any $X \subseteq G$, let $X_{|k} = \{x_{|k} \mid x \in X\}$.

In the literature, one may find other definitions. Usually, finite groups are considered to be profinite. Throughout this paper, profinite group means infinite profinite group.

Let us make the following remark. If we consider only an infinite subsequence of (G_n) and the compositions of the corresponding homomorphisms, we obtain a new profinite group that is isomorphic to the original one (and we obtain the isomorphism, as well). Applying these subsequences will be very useful: without changing group theoretic properties, we will be able to change other properties.

From now on, we follow [2] in terminology. We introduce the natural metric on G:

Definition. *If $x, y \in G$ such that $x \neq y$, then let $d(x, y) = 1/|G_k|$, if $x_{|j} = y_{|j}$ for all $j < k$ and $x_{|k} \neq y_{|k}$. Let $d(x, x) = 0$.*

One can readily check that this metric makes G a Polish space. Although this metric can be changed, when we switch to a subsequence of (G_n), the topology remains the same.

1 Box-dimension

Let $|S|$ denote the cardinality of set S.

Definition. *Assume $X \subseteq G$. Then let*

$$\underline{\dim}_{\mathrm{B}}(X) = \liminf_{k\to\infty} \frac{\log |X_{|k}|}{\log |G_{|k}|}; \quad \overline{\dim}_{\mathrm{B}}(X) = \limsup_{k\to\infty} \frac{\log |X_{|k}|}{\log |G_{|k}|}$$

the lower and upper box-dimension of X, respectively. If $\underline{\dim}_{\mathrm{B}}(X) = \overline{\dim}_{\mathrm{B}}(X)$, then we say that $\dim_{\mathrm{B}}(X)$ exists and its value is $\underline{\dim}_{\mathrm{B}}(X) = \overline{\dim}_{\mathrm{B}}(X)$.

Throughout this section, dimension means box-dimension.

If G is a given profinite group, we can consider its n-th power G^n. We can define the lower and upper box-dimensions for the subsets of $G^n = G \times \ldots \times G$ as above, except for the denominator, in which we still divide by $\log |G_{|k}|$. Hence for example, $\dim_{\mathrm{B}}(G^n) = n$. The metric of G^n is defined as follows:

$$d_n((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max(d(x_1, y_1), \ldots, d(x_n, y_n)).$$

Let $X, Y \subseteq G$. One can readily check that $\underline{\dim}_{\mathrm{B}}(X)+\underline{\dim}_{\mathrm{B}}(Y) \leq \underline{\dim}_{\mathrm{B}}(X\times Y)$, $\overline{\dim}_{\mathrm{B}}(X) + \overline{\dim}_{\mathrm{B}}(Y) \geq \overline{\dim}_{\mathrm{B}}(X \times Y)$. Hence if $\dim_{\mathrm{B}}(X), \dim_{\mathrm{B}}(Y)$ exist, then $\dim_{\mathrm{B}}(X) + \dim_{\mathrm{B}}(Y) = \dim_{\mathrm{B}}(X \times Y)$.

Lemma 1.1. *If the interior of $X \subseteq G$ is nonempty, then $\dim_{\mathrm{B}}(X) = 1$. If the interior of $X \subseteq G^n$ is nonempty, then $\dim_{\mathrm{B}}(X) = n$.*

Proof. It is sufficient to verify the 1-dimensional statement. Let $x = (x_1, x_2, \ldots)$ be an interior point of X. Then for some $k \in \mathbb{N}$, $(x_1, \ldots, x_k, g_{k+1}, g_{k+2}, \ldots) \in X$ for all $g_{k+1} \in G_{k+1}, g_{k+2} \in G_{k+2}, \ldots$ such that $(x_1, \ldots, x_k, g_{k+1}, g_{k+2}, \ldots) \in G$. Then for all $l \geq k$, $|X_{|l}| \geq |G_{|l}|/|G_{|k}|$. Hence $\log |X_{|l}|/ \log |G_{|l}| \to 1$, since $|G_{|l}| \to \infty$. $\square$

If $X \subseteq G$, then let

$$X^n_* = \{(x_1, \ldots, x_n) \in X^n | i \neq j \Rightarrow x_i \neq x_j\}.$$

Let $F_n : G^{n+1} \to G^n$ be the following function:

$$F_n(x_1, \ldots, x_n, g) = (x_1 g, \ldots, x_n g).$$

It is easy to see that $d(F_n(x), F_n(y)) \leq d(x, y)$, which implies that F is continuous. One can see that F_n cannot increase the lower and upper dimensions, since $|X_{|k}| \geq |F(X)_{|k}|$.

Lemma 1.2. *Let $X \subseteq G$ be such that $F_n(X^n \times G) \cap P_*^n = \emptyset$. Then for all $g \in G$, $|Xg \cap P| \leq n-1$. If $|P| = 2^\omega$, then G cannot be covered with less than continuum many right-translates of X. (Where right-translate of X means Xg.)*

Proof. Let us suppose that the first claim is false. Then $x_1 g = y_1, \dots, x_n g = y_n$ are different elements of P, where $x_1, \dots, x_n \in X$. Hence $F_n(x_1, \dots, x_n, g) \in P_*^n$ and this is a contradiction. The second statement easily follows from the first one. □

Lemma 1.3. *For some fixed $n \geq 2$, let $F \subseteq G^n$ be compact, nowhere dense. Then there exists a perfect subset $P \subseteq G$ such that $|P| = 2^\omega$ and $F \cap P_*^n = \emptyset$.*

Proof. First, define a decreasing sequence (U_k) of open sets.

Let $U_0 = G$, which is open. In general, the open set U_k is the union of n^k open components. The components of U_{k+1} will be chosen such that they will satisfy the following properties:

(i) each component of U_k contains n open components of U_{k+1};

(ii) every component of U_{k+1} has diameter at most $1/2^k$;

(iii) the closure of every component of U_{k+1} is in U_k;

(iv) denote the components of U_{k+1} by $V_1, \dots, V_{n^{k+1}}$, if $x_1, \dots, x_n$ are elements of pairwise different $V_{j_1}, \dots, V_{j_n}$, then $(x_1, \dots, x_n) \notin F$.

We prove that $P = \cap_{k=0}^{\infty} U_k$ has all the properties we stated in the Lemma. First $|P| = 2^\omega$ holds by (i), (iii) and the assumption $n \geq 2$. By contradiction, assume $(x_1, \dots, x_n) \in F \cap P_*^n$. Choose k such that $\min_{1 \leq i \neq j \leq n} d(x_i, x_j) > 1/2^k$. Now by (ii), we see that $x_1, \dots, x_n$ lie in pairwise different components $V_{j_1}, \dots, V_{j_n}$ of U_{k+1}. This contradicts (iv).

Now we are left to show that one may construct the set U_{k+1} starting out from U_k such that the prescribed properties simultaneously hold. First, take some $V_1, \dots, V_{n^{k+1}}$ satisfying (i), (ii), (iii). Next shrink these subsets as follows, this will not affect these first three properties. Let $\pi : \{1, \dots, n\} \to \{1, \dots, n^{k+1}\}$ be an injective function. Let $V_1' \subseteq V_1, \dots, V_{n^{k+1}}' \subseteq V_{n^{k+1}}$ such that $V_{\pi(1)}' \times \dots \times V_{\pi(n)}' \cap F = \emptyset$ (there exist such subsets, because F is not dense in $V_{\pi(1)} \times \dots \times V_{\pi(n)}$). We see that if $x_1 \in V_{\pi(1)}', \dots, x_n \in V_{\pi(n)}'$, then $(x_1, \dots, x_n) \notin F$. Now we change the notation, writing again $V_1, \dots, V_{n^{k+1}}$ in place of $V_1', \dots, V_{n^{k+1}}'$ and repeat this shrinking operation for every injective function $\pi : \{1, \dots, n\} \to \{1, \dots, n^{k+1}\}$. Finally, we obtain a subset $U_{k+1} = V_1 \cup \dots \cup V_{n^{k+1}}$ of U_k satisfying (iv). □

Lemma 1.4. *Let $X \subseteq G$ be compact such that $\overline{\dim}_{\mathrm{B}}(X) < 1$. Then G cannot be covered with less than continuum many right-translates of X.*

Proof. Let $n \geq 2$ be such that $n\overline{\dim}_{\mathrm{B}}(X) < n-1$, which implies $\overline{\dim}_{\mathrm{B}}(X^n) \leq n\overline{\dim}_{\mathrm{B}}(X) < n-1$. Since $X^n \times G \subseteq G^{n+1}$ and $\overline{\dim}_{\mathrm{B}}(X^n \times G) < n$, $\overline{\dim}_{\mathrm{B}}(F_n(X^n \times G)) < n$. By Lemma 1.1, the interior of the compact set $F_n(X^n \times G) \subseteq G^n$ is empty, then $F_n(X^n \times G)$ is nowhere dense. Then Lemma 1.3 finishes the proof. □

By switching to a well-chosen subsequence of (G_n), we can strengthen our theorem to lower dimension.

Theorem 1.5. *Let $X \subseteq G$ be compact such that $\underline{\dim}_{\mathrm{B}}(X) < 1$. Then G cannot be covered with less than continuum many right-translates of X.*

Proof. Let us choose a subsequence of G_n. The obtained group, denoted by $\widetilde{G}$ is isomorphic to G. Let X be a given compact subset of G such that $\underline{\dim}_{\mathrm{B}}(X) < 1$. By definition, for a convenient subsequence, $\widetilde{\dim_{\mathrm{B}}}(\widetilde{X}) = \underline{\dim}_{\mathrm{B}}(X) < 1$ holds. Since less than continuum many right-translates of $\widetilde{X}$ cannot cover $\widetilde{G}$, less than continuum many right-translates of X cannot cover G. $\square$

2 Hausdorff-dimension

We define the Hausdorff-dimension following [2].

Definition. *Let $X \subseteq G$. Then*

$$\mu_\delta^s(X) = \inf\left\{\sum_{n=1}^{\infty}(\mathrm{diam}(A_n))^s \mid \bigcup_{n=1}^{\infty} A_n \supseteq X, \mathrm{diam}(A_n) < \delta\right\},$$

$$\mu^s(X) = \lim_{\delta \to 0} \mu_\delta^s(X).$$

Furthermore,

$$\dim_{\mathrm{H}}(X) = \sup\{t \mid \mu^t(X) = \infty\}(= \inf\{t \mid \mu^t(X) = 0\})$$

is the Hausdorff-dimension of X.

In this section, dimension means Hausdorff-dimension. Our aim is to prove that in an appropriate model, there exists a compact subset $C \subseteq G$ with $\dim_{\mathrm{H}}(C) = 0$ such that G is the union of less than continuum many translates of C. We will use the following notation: $N_j \lhd G$ is the normal subgroup of G consisting of elements, whose j-th coordinate is 1 (the unit element of the corresponding group), that is, $N_j = \{x \in G : x_{|j} = 1\}$.

Lemma 2.1. *Let $n, k \in \mathbb{N}$. Then there exists a compact set $K_{n,k} \subseteq G$ with the following properties:*

(i) it is the union of closed subsets F_i such that $\sum_i \mathrm{diam}(F_i)^{1/k} < 1/k$;

(ii) for arbitrary $x_1, \ldots, x_n \in G$ there exists $g \in G$ such that $x_1 g, \ldots, x_n g \in K_{n,k}$;

(iii) $K_{n,k}$ is the union of a normal subgroup N_j and some of its cosets.

Proof. Fix $k \in \mathbb{N}$. We will construct $K_{n,k}$ by induction on n. For $n = 1$, it is enough to put $K_{1,k} = N_j$ for a sufficiently large j.

Now assume $K_{n-1,k}$ is constructed. We will obtain $K_{n,k}$ by adding some subsets to $K_{n-1,k}$. By (iii), $K_{n-1,k}$ consists of a normal subgroup N_j and some of its cosets. Further by (ii), for some closed subsets F_i and $\varepsilon > 0$, $K_{n-1,k} = \cup_i F_i$ and

$$\sum_i \mathrm{diam}(F_i)^{\frac{1}{k}} < \frac{1}{k} - \varepsilon.$$

Take an element from each coset of N_j, denote them by $h_1, \dots, h_m$. Then take some closed balls $B_1, \dots, B_m$ such that

(a) $h_1 \in B_1, \dots, h_m \in B_m$;

(b)
$$\sum_{i=1}^{m} \text{diam}(B_i)^{\frac{1}{k}} < \varepsilon;$$

(c) the balls $B_1, \dots, B_m$ are all cosets of a normal subgroup $N_{j'}$ for some $j' > j$.

We claim that $K_{n,k} = \cup_i B_i \cup K_{n-1,k}$ has all the properties we need. The induction hypothesis, (a) and (c) give (i) and (iii) immediately. As for (ii), let $x_1, \dots, x_n$ be arbitrary elements of G. By the induction hypothesis, for elements $x_1, \dots, x_{n-1}$, there is a $g \in G$ such that $x_1 g, \dots, x_{n-1} g \in K_{n-1,k}$. Then $x_n g$ is in one of the cosets corresponding to N_j, which implies that for an appropriate element $g' \in N_j$, $x_n g g' = h_i \in B_i$. Furthermore, multiplication with g' from the right ($x \mapsto xg'$) maps each coset of N_j to itself, hence gg' is an appropriate element: $x_1 gg', \dots, x_n gg' \in K_{n,k}$. □

Corollary 2.2. *If $n, k \in \mathbb{N}$, N_j is a given normal subgroup and H is a coset of N_j, then there exists a subset $K^H_{n,k} \subseteq H$ satisfying the following properties:*

1. *it is the union of closed subsets F_i such that $\sum_i \text{diam}(F_i)^{1/k} < 1/k$;*
2. *for arbitrary $x_1, \dots, x_n \in H$ there exists a $g \in N_j$ such that $x_1 g, \dots, x_n g \in K^H_{n,k}$;*
3. *$K^H_{n,k}$ is the union of some cosets corresponding to a normal subgroup $N_{j'}$ ($j' > j$).*

Proof. Let us take $K_{n,k}$ from Lemma 2.1. Denoting by $H_1, \dots, H_m$ the cosets of N_j, there exist $h_1, \dots, h_m \in G$ such that $H_1 h_1 = \dots = H_m h_m = H$. Let

$$K^H_{n,k} = \bigcup_{i=1}^{m} (K_{n,k} \cap H_i) h_i.$$

One can easily check that the defined set satisfies the conditions. □

Definition. *Let $f : \mathbb{N} \to \mathbb{N}$. Then we call a set $\prod_{n=1}^{\infty} A_n$ f-slalom, if $|A_n| \leq f(n)$ for all $n \in \mathbb{N}$. Let f and an f-slalom $\prod_{n=1}^{\infty} A_n$ be given. We say that $g : \mathbb{N} \to \mathbb{N}$ is a sub-slalom-size, if $g(n) \leq f(n)$ holds for all $n \in \mathbb{N}$. We call a set $\prod_{n=1}^{\infty} B_n$ g-sub-slalom, if it is a g-slalom and $B_n \subseteq A_n$ for all $n \in \mathbb{N}$.*

Let $f, g : \mathbb{N} \to \mathbb{N}$, where $g(i) < f(i)$. Assume that $f(i)$ and $g(i)$ tend to infinity. Recall that $\mathcal{M}$ is a model for ZFC with the further consistent assumption $cof(\mathcal{N}) < 2^{\omega}$. In $\mathcal{M}$, every f-slalom can be covered with less than continuum many g-sub-slaloms: the method is described in [5, page $9 - 10$.], where the authors use [7, Theorem 2.10.] and [3, Theorem 2.3.9. and page 388.]

Our plan is the following. We define an f-slalom that corresponds to G and introduce a g sub-slalom-size that tends to infinity. Using the sets $K_{n,k}$, we construct a zero-dimensional compact subset $C \subseteq G$ such that any g-sub-slalom can be right-translated into C.

2.1 Construction of C.

Consider the set $\widetilde{K}_{1,1} = K_{1,1}$ given by Lemma 2.1. Assume that it consists of a single normal subgroup N_j.

Assume that we are on the m-th level and we have already defined the subset $\widetilde{K}_{(m-1)!,(m-1)!}$ that consists of some cosets of the same size (corresponding to the same normal subgroup). Denote one of these cosets by H. Then $\widetilde{K}_{(m-1)!,(m-1)!}$ consists of some appropriate left-translates of H: $H = H_1, \dots, H_{S(m-1)}$. Hence for appropriate elements $1 = h_1, \dots, h_{S(m-1)}$, $H = h_1 H_1 = \dots = h_{S(m-1)} H_{S(m-1)}$. In H, take the subset $K^H_{m!,m!S(m-1)}$ given by Corollary 2.2. Then apply every h_i^{-1} and let

$$\widetilde{K}_{m!,m!} = \bigcup_{i=1}^{S(m-1)} h_i^{-1} K^H_{m!,m!S(m-1)}.$$

By this induction, we obtain a set $\widetilde{K}_{m!,m!}$ for all m.

Then $(\widetilde{K}_{m!,m!})$ is a sequence of decreasing compact sets. Let

$$C = \bigcap_{m=1}^{\infty} \widetilde{K}_{m!,m!}.$$

Lemma 2.3. $\dim_{\mathrm{H}}(C) = 0$.

Proof. By the definition of $K^H_{m!,m!S(m-1)}$, there exist some closed sets F_i such that $K^H_{m!,m!S(m-1)} = \cup_i F_i$ and

$$\sum_i \operatorname{diam}(F_i)^{\frac{1}{m!S(m-1)}} < \frac{1}{m!S(m-1)}.$$

Since $\widetilde{K}_{m!,m!}$ is the union of $S(m-1)$ copies of $K^H_{m!,m!S(m-1)}$, it is covered with the union of closed sets F_i such that

$$\sum_i \operatorname{diam}(F_i)^{\frac{1}{m!S(m-1)}} < \frac{1}{m!}.$$

Then, by increasing the exponent ($\operatorname{diam}(G) \leq 1$),

$$\sum_i \operatorname{diam}(F_i)^{\frac{1}{m!}} < \frac{1}{m!},$$

which implies that C is zero-dimensional. □

2.2 Defining the slalom corresponding to G.

Consider the following subsequence of (G_n). Let N_{m_k} be a strictly decreasing sequence of the normal subgroups such that $\widetilde{K}_{m!,m!}$ consists of cosets corresponding to N_{m_k}. We can switch to any infinite subsequence of (G_n). Choose a subsequence (denoted by (G_m)) such that the following two conditions hold: $\widetilde{K}_{m!,m!}$ consists of cosets corresponding to N_m; $|G_m/G_{m-1}| > m$ (where G_0 is the trivial group). From this point on, we use this sequence, which defines G, as well. Recall that the topology is the same, however the metric could change (let the new metric be d'). Let $f(m) = |G_m/G_{m-1}|$ and let $g(m) = m$. The f-slalom corresponding to G is the following: $\prod_{m=1}^{\infty} G_m/G_{m-1}$. It is indeed an f-slalom and it represents G as follows: the elements of G are in natural bijection with the elements of the product of the quotient maps φ_m $(m = 1, 2, \ldots)$.

Lemma 2.4. *Let Z be a g-sub-slalom. There exists a $\overline{y} \in G$ such that $Z\overline{y} \subseteq C$.*

Proof. Recall the notations of the construction. Let $\widetilde{K}_{(m-1)!,(m-1)!}$ consists of the cosets $H = H_1, \ldots, H_{S(m-1)}$ that correspond to N_{m-1}. Let $1 = h_1, \ldots, h_{S(m-1)} \in G$ be such that $H = h_1H_1 = \ldots = h_{S(m-1)}H_{S(m-1)}$.

First, we prove by induction that for all m there exists a $y_m \in G$ such that $(Zy)_{|m} \subseteq (\widetilde{K}_{m!,m!})_{|m}$. This is clear for $m = 1$, since $|Z_{|1}| = 1$ and $\widetilde{K}_{1!,1!}$ is nonempty. Suppose that it holds for $m-1$. For all i, $|Z_{|i}| \leq i!$. By induction, there exists $x \in G$ such that

$$(Zx)_{|m-1} \subseteq (\widetilde{K}_{(m-1)!,(m-1)!})_{|m-1} = (H_1)_{|m-1} \cup \ldots \cup (H_{S(m-1)})_{|m-1}.$$

Since $H_1, \ldots, H_{S(m-1)}$ are cosets of N_{m-1}, if they contain a subset restricted to G_{m-1}, then so do in G. Therefore $(Zx)_{|m} \subseteq (H_1)_{|m} \cup \ldots \cup (H_{S(m-1)})_{|m}$. For all $b \in Z$, we take a left-translate of $(bx)_{|m} \in (Zx)_{|m}$ that is in the coset $H_{|m}$ the following way: for each element $(bx)_{|m} \in (H_i)_{|m}$, consider $(h_ibx)_{|m}$. The number of elements of the form $(bx)_{|m}$ is at most $m!$, since Z is a g-slalom. We left-translated all of these elements $(bx)_{|m}$ by exactly one element h_i. Therefore we have at most $m!$ elements of the form $(h_ibx)_{|m}$ in $H_{|m}$. Then we can right-translate all element of the form $(h_ibx)_{|m}$ into $(K^H_{m!,m!S(m-1)})_{|m}$ by an appropriate element $h \in G$: continue each element of the form $(h_ibx)_{|m}$ to an element of G, these continuations can be right-translated into $K^H_{m!,m!S(m-1)}$, by its definition. Then xh is a suitable element y_m, because for any $b \in Z$ $((bx)_{|m} \in (H_i)_{|m})$

$$(bxh)_{|m} = (h_i^{-1}h_ibxh)_{|m} \in (h_i^{-1}K^H_{m!,m!S(m-1)})_{|m} \subseteq (\widetilde{K}_{m!,m!})_{|m},$$

which completes the proof of the first claim.

For all m, consider the element $y_m \in G$ such that $(Zy_m)_{|m} \subseteq (\widetilde{K}_{m!,m!})_{|m}$. Because of the compactness of G, (y_m) has a convergent subsequence. Let its limit be $\overline{y}$. We claim that $Z\overline{y} \subseteq C$. We will prove it by contradiction. Suppose that for an element $b \in Z$, $b\overline{y} \notin C$. Let $d'(b\overline{y}, C) = D > 0$. If $m > m_0$, then $d'(by_m, b\overline{y}) < D/2$. Furthermore, if $m > m_1$, then $\widetilde{K}_{m!,m!} \subseteq U_{D/2}(C)$, where $U_\varepsilon(C) = \{x \in G \mid d'(x, C) < \varepsilon\}$. Since $(by_m)_{|m} \in (\widetilde{K}_{m!,m!})_{|m}$, $by_m \in \widetilde{K}_{m!,m!}$ (because of the fact that $\widetilde{K}_{m!,m!}$ is the union of complete cosets corresponding to

N_m, if it contains an element restricted to G_m, then so do in G). Hence $d'(by_m, C) < D/2$. If $m > m_0, m_1$, then

$$d'(b\overline{y}, C) \leq d'(by_m, b\overline{y}) + d'(by_m, C) < D,$$

which is a contradiction. □

Theorem 2.5. *Let G be a profinite group. In $\mathcal{M}$, there exists a compact subset X of Hausdorff-dimension 0 such that G can be covered with less than continuum many right-translates of X.*

Proof. By Lemma 2.3, C is zero-dimensional and by Lemma 2.4, every g-sub-slalom can be covered with an appropriate right-translate of C. Since G is the union of less than continuum many g-sub-slaloms in $\mathcal{M}$, G is the union of less than continuum many right-translates of C. □

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